# Zero-dimensional spaces and subspaces in computable analysis

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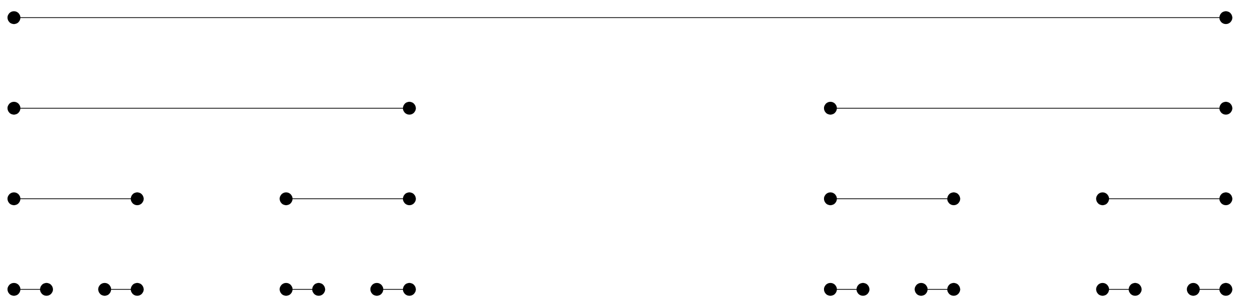
## Construction of triadic Cantor set $C$:



$C = \{\sum_{i=1}^{\infty} a_i 3^{-i} \,\big|\, a_i \in \{0, 2\} \text{ for all } i\}$

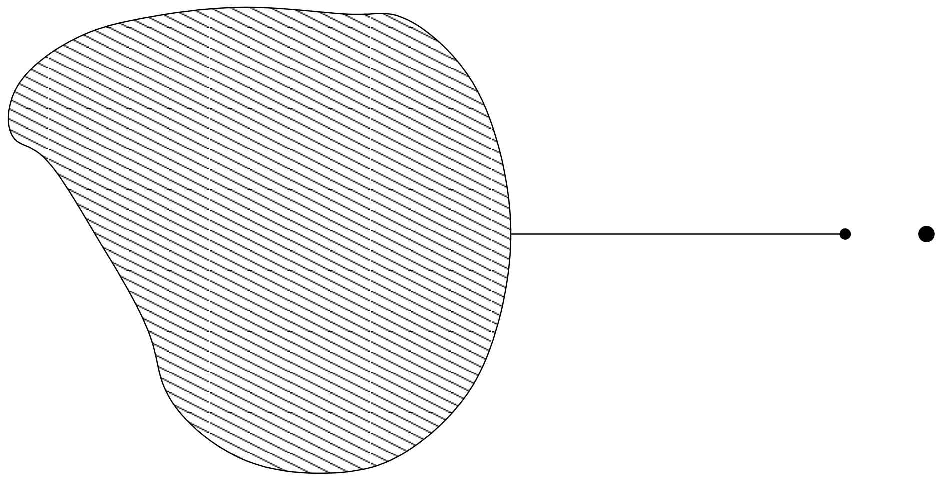
Cantor's (onto) mapping:

$$f : C \to [0, 1], \sum_{i=1}^{\infty} a_i 3^{-i} \mapsto \sum_{i=1}^{\infty} b_i 2^{-i}$$

where $b_i = \frac{a_i}{2}$

$C$ has same cardinality as $[0, 1]$.

## Open and closed sets in subspaces $Y \subseteq \mathbb{R}^n$:



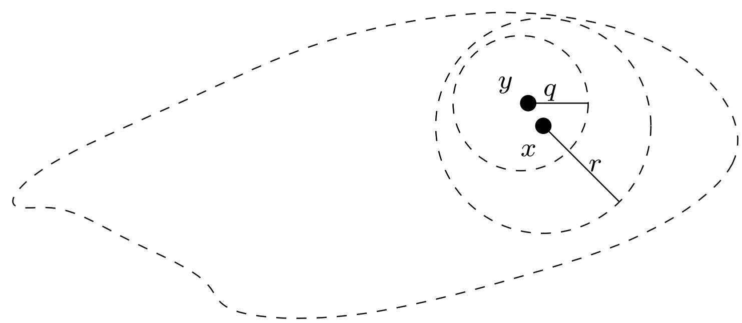
**open** subset $U$ of $Y \subseteq \mathbb{R}^n$: $U$ contains a ball $Y \cap B(x; r)$ about each point $x \in U$;

**closed** subset $A$ of $Y$: $Y \setminus A$ is open in $Y$

$(U_i)_{i \geq 1}$ a **basis** for open sets: every $U_i$ is open, & for any $x \in Y$ and open $U \ni x$ there exists $i$ with $x \in U_i \subseteq U$

If $Z \subseteq Y$ and $(U_i)_{i \geq 1}$ a basis for open sets in $Y$, $(U_i \cap Z)_{i \geq 1}$ a basis for open sets in $Z$.

## Basis $\alpha$ for $\mathbb{R}^n$:




$d(x,y) + q < r \implies B(y;q) \subseteq B(x;r)$

$\mathbb{N} = \{0, 1, 2, \ldots\}$

Invertible pairing function $\langle \cdot, \cdot \rangle : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$;

$\nu_{\mathbb{Q}} : \mathbb{N} \to \mathbb{Q}, \langle 2a + i, b \rangle \mapsto \frac{(-1)^i a}{b+1}, \quad \nu_{\mathbb{Q}^+} : \mathbb{N} \to \mathbb{Q}^+, \langle a, b \rangle \mapsto \frac{a+1}{b+1}$

basis $\alpha : \mathbb{N} \to \Sigma^0_1(X), \langle i, j \rangle \mapsto B(\nu(i); \nu_{\mathbb{Q}^+}(j))$

Another example: closed-and-open (**clopen**) subsets of $Y$

$Y = C$ **zero-dimensional** – any $x \in Y$ & open $U \ni x$ have some clopen $V$ with $x \in V \subseteq U$.

Given basis $\alpha$, represent open sets by infinite sequences $p = (p_0, p_1, \dots)$ over $\mathbb{N}$:

$\delta_{\Sigma^0_1} : \mathbb{N}^{\mathbb{N}} \to \mathbf{\Sigma}^0_1(X), p \mapsto \cup\{\alpha(p_i - 1) \,\big|\, i \in \mathbb{N}, p_i \geq 1\}$

$\delta_{\Pi^0_1}(p) = X \setminus \delta_{\Sigma^0_1}(p), \quad \delta_{\Delta^0_1}\langle p, q\rangle = \delta_{\Sigma^0_1}(p) = \delta_{\Pi^0_1}(q)$

$\delta :\subseteq \mathbb{N}^{\mathbb{N}} \to X,\ p \in \delta^{-1}\{x\} \iff \{p_i \,\big|\, p_i \geq 1\} = \{a + 1 \,\big|\, \alpha(a) \ni x\}$;

$p$ a padded list of all basis elements $\ni x$.

$\sqsubset \subseteq \mathbb{N}^2$ a **formal inclusion** if $a \sqsubset b$ implies $\alpha(a) \subseteq \beta(b)$

$\langle i, j\rangle \sqsubset \langle k, l\rangle :\iff d(\nu(i), \nu(k)) + \nu_{\mathbb{Q}^+}(j) < \nu_{\mathbb{Q}^+}(l)$

Then (for $\beta = \alpha$)

$$(\forall b)(\forall x \in X)(\exists a)\,(x \in \beta(b) \implies x \in \alpha(a) \wedge a \sqsubset b) \quad (1)$$

$F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}, p \mapsto q$ **computable** if there exists an algorithm to read finite prefixes of $p$ and output $q$ left-to-right.

Equivalent statements of zero-dimensionality (for $X$ separable metrizable):

(A) there exists a basis $(V_i)_1^\infty$ of clopen sets;

(B) for any closed $A \subseteq X$ and $p \in X \setminus A$, $\emptyset$ is a partition between $p$ and $A$;

(C) for any disjoint closed $A, B \subseteq X$, $\emptyset$ is a partition between $A$ and $B$;

(D) for any open cover $(U_i)_1^r$ of $X$, there exists an open disjoint shrinking $(V_i)_1^r$.

$P$ a *partition between $A$ and $B$* if there exist open $U, V$ with $A \subseteq U$, $B \subseteq V$ and $U \dot{\cup} V = X \setminus P$

## Effective zero-dimensionality for $X$:

(A)' there exist computable basis $b : \mathbb{N} \to \mathbf{\Delta}^0_1(X)$ and computably enumerable formal inclusion $\sqsubset'$ of $b$ with respect to $\alpha$ with property (1);

(B)' $M :\subseteq X \times \mathbf{\Sigma}^0_1(X) \rightrightarrows \mathbf{\Delta}^0_1(X), (x, U) \mapsto \{W \mid x \in W \subseteq U\}$ ($\mathrm{dom}\, M = \{(x, U) \mid x \in U\}$) is computable;

(C)' $N :\subseteq \mathbf{\Pi}^0_1(X)^2 \rightrightarrows \mathbf{\Delta}^0_1(X), (A, B) \mapsto \{W \mid A \subseteq W \wedge B \subseteq X \setminus W\}$ ($\mathrm{dom}\, N = \{(A, B) \mid A \cap B = \emptyset\}$) is computable;

(D)' $C^* :\subseteq \mathbf{\Sigma}^0_1(X)^* \rightrightarrows \mathbf{\Sigma}^0_1(X)^*, (U_i)_{i<k} \mapsto \{(V_i)_{i<k} \mid (\forall i, j)(i \neq j \implies V_i \cap V_j = \emptyset), V_i \subseteq U_i, \cup_i V_i = X\}$ ($\mathrm{dom}\, C^* = \{(U_i)_{i<k} \mid k \in \mathbb{N} \wedge \cup_i U_i = X\}$) is computable.

**Proof of (D)' $\implies$ (C)':** $(U_i)_{i<2} = (X \setminus A, X \setminus B)$ covers $X$ and $(W_i)_{i<2} \in C^*((U_i)_i)$ implies $W_1 = X \setminus W_0 \supseteq A$, $X \setminus W_1 \supseteq B$, so $W_1 \in N(A, B)$, for any pairwise disjoint $A, B \in \mathbf{\Pi}^0_1(X)$. We have $\delta_{\mathbf{\Delta}^0_1(X)}$-information on $W_1$.

## Proof of (A)' $\Longrightarrow$ (D)':

In fact $\tilde{C} : \mathbf{\Sigma}^0_1(X)^{\mathbb{N}} \rightrightarrows \mathbf{\Sigma}^0_1(X)^{\mathbb{N}}, (U_i)_i \mapsto \{(W_i)_i \,\big|\, (\forall i,j)(i \neq j \implies W_i \cap W_j = \emptyset), W_i \subseteq U_i, \cup_i W_i = \cup_i U_i\}$ is computable.

Proof follows Kuratowski (1966, §26.II, Thm 1). Given $\langle p^{(0)}, p^{(1)}, \ldots \rangle \in (\delta^{\omega}_{\Sigma^0_1})^{-1}\{(U_i)_i\}$ and $i \in \mathbb{N}$, list:

$$0 \text{ for each } j \;\text{ s.t. } p^{(i)}_j = 0,$$
$$a+1 \text{ for any } a,j \;\text{ s.t. } p^{(i)}_j \geq 1 \wedge (a \sqsubset' p^{(i)}_j - 1).$$

Properties of $\sqsubset'$ and $\alpha$ (namely, $\operatorname{im} \alpha \not\ni \emptyset$) imply output is infinite; let $W_{i,j} :=$ ($\emptyset$ if $e = 0$; $b(a)$ if $e = a+1$) for $j^{\text{th}}$ output $e$. By properties of $\sqsubset'$, $U_i = \cup_j W_{i,j}$.

Compute $\delta_{\Delta^0_1}$-names for $W^*_{i,j} := W_{i,j} \setminus \left(\cup_{\langle k,l \rangle < \langle i,j \rangle} W_{k,l}\right)$ from information on $b$. Then $V_i := \cup_j W^*_{i,j}$ ($\subseteq U_i$) are disjoint with $\cup_i U_i = \cup_i V_i$ and a $\delta^{\omega}_{\Sigma^0_1}$-name of $(V_i)_i$ is computable.

## Proof of (B)' $\Longrightarrow$ (A)':

**Lemma 1.** *For computable metric spaces $(X, d, \nu)$, $(Z, d', \nu')$ and strictly normed Cauchy representation $\delta_Z$ of $Z$, the computable dense sequence $z_i := \nu'(i)$ ($i \in \mathbb{N}$) satisfies*

$$\cup_{i\in\mathbb{N}} u(z_i) = \cup_{z\in Z} u(z)$$

*for any $(\delta_Z, \delta_{\Sigma^0_1(X)})$-continuous $u : Z \to \Sigma^0_1(X)$.*

$M^\circ : X \times \mathbb{N} \rightrightarrows \Delta^0_1(X), (x, i) \mapsto \{W \,\big|\, x \in W \subseteq B(x; 2^{-i})\}$,
$E : X \times \mathbb{N} \to \Sigma^0_1(X), (x, i) \mapsto B(x; 2^{-i})$.

Briefly, $M^\circ = M \circ (\pi_1, E)$. With computable realiser $G :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ of $M^\circ$ and inclusion $\iota : \Delta^0_1(X) \to \Sigma^0_1(X)$, apply Lemma 1:

$$\begin{array}{ccc} \mathbb{N}^\mathbb{N} & \xrightarrow{G} & \mathbb{N}^\mathbb{N} \\ \downarrow & & \downarrow \delta_{\Delta^0_1} \\ X \times \mathbb{N} & \rightrightarrows & \Delta^0_1(X) \end{array}$$

$u^{(i)} = (\iota \circ \delta_{\Delta^0_1} \circ G)\langle \cdot, i.0^\omega\rangle : \operatorname{dom}\delta \to \Sigma^0_1(X)$ ($i \in \mathbb{N}$),
$Z := \operatorname{dom}\delta$ with a standard enumeration $(p^{(i)})_i$ of $\{w.w^\omega_{|w|-1} \,\big|\, w \in A\}$ as $\nu'$, where

$$\begin{aligned} A &:= \{w \in \mathbb{N}^* \,\big|\, (\forall i, j < |w|) d(\nu(w_i), \nu(w_j)) < 2^{-\min\{i,j\}}\} \\ &= \{w \in \mathbb{N}^* \,\big|\, w.\mathbb{B} \cap \operatorname{dom}\delta \neq \emptyset\}. \end{aligned}$$

$\cup_{j\in\mathbb{N}}\, u^{(i)}(p^{(j)}) = \cup_{p\in\mathrm{dom}\,\delta} u^{(i)}(p) = X$ for every $i \in \mathbb{N}$.

$$b : \mathbb{N} \to \mathbf{\Delta}^0_1(X), \langle i, j\rangle \mapsto u^{(i)}(p^{(j)});$$

note $(\forall x \in X)(\forall i)(\exists B \in \mathrm{im}\, b)(x \in B \wedge \mathrm{diam}\, B < (i+1)^{-1})$, hence $b$ a basis.

We also define relation $\sqsubset' \subseteq \mathbb{N}^2$:

$$\begin{aligned}\langle i, j\rangle \sqsubset' \langle n, r\rangle :\iff\ & d(\delta(p^{(i)}), \nu(n)) + 2^{-i} < \nu_{\mathbb{Q}^+}(r)\\ \iff\ & (\exists k) d(\nu(p^{(j)}_k), \nu(n)) + 2^{-k} + 2^{-i} < \nu_{\mathbb{Q}^+}(r).\end{aligned}$$

$\sqsubset'$ is c.e. and a formal inclusion of $b$ w.r.t. $\alpha$. Finally, $\sqsubset'$ satisfies the property (1).

## Zero-dimensional subsets

Fix class $\mathcal{Y} \subseteq \mathbb{P}(X)$ of zero-dim or empty sets with cardinality $|\mathcal{Y}| \leq 2^{\aleph_0}$. Consider following:

(A)" $\breve{B}'' : \mathcal{Y} \rightrightarrows (\mathbf{\Sigma}_1^0(X)^2)^{\mathbb{N}} \times (\mathbb{N}^2)^{\mathbb{N}}$ is computable

(B)" $M'' :\subseteq X \times \mathbf{\Sigma}_1^0(X) \times \mathcal{Y} \rightrightarrows \mathbf{\Sigma}_1^0(X)^2$ is computable

(C)" $N'' :\subseteq \mathbf{\Pi}_1^0(X)^2 \times \mathcal{Y} \rightrightarrows \mathbf{\Sigma}_1^0(X)^2$ is computable

(D)" $\breve{C}^{\omega\circ} :\subseteq \mathbf{\Sigma}_1^0(X)^{\mathbb{N}} \times \mathcal{Y} \rightrightarrows \mathbf{\Sigma}_1^0(X)^{\mathbb{N}}$ is computable

$\operatorname{dom} \breve{C}^{\omega\circ} = \{((U_i)_i, Y) \mid \cup_i U_i \supseteq Y\}$, $\operatorname{dom} M'' = \{(x, U, Y) \mid x \in U\}$, $\operatorname{dom} N'' = \{(A, B, Y) \mid A \cap B = \emptyset\}$,

$$\breve{B}''(Y) := \Big\{((U_i, V_i)_i, (a_i, b_i)_i) \,\Big|\, (U_i)_i \text{ basis for } \mathcal{T}_X,\ (\forall i) Y \subseteq U_i \dot\cup V_i \text{ and } \{(a_k, b_k) \mid k \in \mathbb{N}\} \text{ formal inclusion of } (U_i)_i \text{ w.r.t. } \alpha \text{ with (1)}\Big\},$$

$$\breve{C}^{\omega\circ}((V_i)_i, Y) := \{(W_i)_i \mid (\forall i) W_i \subseteq V_i \wedge \cup_i W_i \supseteq Y \text{ and } (W_i)_i \text{ p.w. disjoint}\},$$

$$M''(x, U, Y) := \{(V, W) \mid x \in V \subseteq U \wedge Y \subseteq V \dot\cup W\},$$

$$N''(A, B, Y) := \{(U, V) : A \subseteq U \wedge B \subseteq V \wedge Y \subseteq U \dot\cup V\}$$

Then $(C)'' \implies (B)'' \implies (A)'' \implies (D)''$.

Now consider $\mathcal{Z}_{\mathsf{c}}(X) := \{K \in \mathcal{K}(X) \mid \dim Y \le 0\}$.

Ideal covers $u, v \in \mathbb{N}^*$ **formally disjoint** if
$(\forall i < |u|)(\forall j < |v|) d(\nu(\pi_1 u_i), \nu(\pi_1 v_j)) > \nu_{\mathbb{Q}^+}(\pi_2 u_i) + \nu_{\mathbb{Q}^+}(\pi_2 v_j)$.

Denote $U\langle u\rangle := \cup_{i<|u|}\alpha(u_i)$,
$p \in (\delta'_{\text{disj-cover}})^{-1}\{K\}$ iff $\{p_i - 1 \mid p_i \ge 1\}$ coincides with

$$\left\{ \langle\langle w^{(0)}\rangle, \ldots, \langle w^{(l-1)}\rangle\rangle \,\middle|\, \begin{array}{l} l \in \mathbb{N}, \cup_{i<l} U\langle w^{(i)}\rangle \supseteq K, \\ (w^{(i)})_{i<l} \subseteq \mathbb{N}^* \text{ p.w. formally disjoint} \end{array} \right\};$$

$p$ padded list of all formally disjoint tuples of ideal covers which together cover $K$.

**Lemma 2.** *In any $X$, $\delta'_{\textit{disj-cover}} \equiv \delta_{\textit{cover}}|^{\mathcal{Z}_{\mathsf{c}}(X)}$.*
**Theorem 3.** *Suppose $X$* effectively locally compact *in the sense $k : \mathbb{N} \times X \rightrightarrows \mathbb{N} \times \mathcal{K}_>(X)$ computable,*
$k(n, x) = \{(a, K) \mid x \in \alpha(a) \wedge \widehat{\alpha}(a) \subseteq K \wedge a \sqsubset n\}$,
$\operatorname{dom} k = \{(n, x) \mid x \in \alpha(n)\}$.

*For $\mathcal{Y} = \{Y \in \Pi^0_1(X) \mid \dim Y \le 0\}$, $\delta_{\mathcal{Y}} := \delta_{\Pi^0_1}|^{\mathcal{Y}}$,*
*$M'' :\subseteq X \times \mathbf{\Sigma}^0_1(X) \times \mathcal{Y} \rightrightarrows \mathbf{\Sigma}^0_1(X)^2$ is computable.*

**Proof:** Let $(x, U, Y) \in \operatorname{dom} M''$, $p \in \delta^{-1}_{\Sigma^0_1}\{U\}$, $q \in \delta^{-1}_{\Pi^0_1}\{Y\}$. Find $a$ and a $\delta_{\text{cover}}$-name for $K \in \mathcal{K}(X)$ s.t. $x \in \alpha(a) \wedge \widehat{\alpha}(a) \subseteq K \wedge (\exists i) p_i \geq 1 \wedge a \sqsubset p_i - 1$. By properties of $\delta_{\Pi^0_1}$, a $\delta_{\text{cover}}$-name of $K \cap Y$ is available.

Given a $\delta'_{\text{disj-cover}}$-name of $K \cap Y$, find $w^{(0)}, \ldots, w^{(l-1)}$ in that name s.t.

$$(\exists i_0)\Big(x \text{ formally incl in } w^{(i_0)} \text{ and } w^{(i_0)} \text{ formally incl in } \alpha(a)\Big)$$

$x \in V := U\langle w^{(i_0)}\rangle \subseteq \alpha(a) \subseteq U$. For $W_0 := \cup_{l>i\neq i_0} U\langle w^{(i)}\rangle$, $W_1 := X \setminus \widehat{\alpha}(a)$, $W := W_0 \cup W_1$ we find $V \subseteq \widehat{\alpha}(a) \subseteq K$ implies $Y = (Y \cap K) \cup (Y \setminus K) \subseteq (V \cup W_0) \cup W_1 = V \cup W$ and $V \cap W = \emptyset$. $\square$

Systematic study of $\breve{B}''$, $M''$, $N''$ variants lacking. Consider e.g. $\mathcal{O}(X) = (\Sigma^0_1(X), \delta_{\mathcal{O}(X)})$ where
$\langle p, q\rangle \in \delta^{-1}_{\mathcal{O}(X)}\{U\} :\iff \delta_{\Sigma^0_1}(p) = U \wedge \delta_{\Sigma^0_1}(q) = X \setminus \overline{U}$,

$$\tilde{M} :\subseteq X \times \Sigma^0_1(X) \times \mathcal{Y} \rightrightarrows \mathcal{O}(X), (x, U, Y) \mapsto \{V \,\big|\, x \in V \subseteq U \wedge Y \cap \partial V = \emptyset\},$$

$\operatorname{dom} \tilde{M} = \{(x, U, Y) \,\big|\, x \in U\}$.

**Corollary 4.** *Denote $c : \mathbb{N} \to \Pi^0_1(X), a \mapsto \overline{\alpha(a)}$. Under conditions of Theorem 3, if $c\big|_{c^{-1}\mathcal{K}(X)}$ computable then $\tilde{M}$ computable.*

# Embedding in Hilbert cube

**Lemma 5.** *(Weihrauch [**?**, Thm 15]) In a computable metric space $X$, $U :\subseteq \Pi^0_1(X)^2 \rightrightarrows C(X,[0,1])$,*

$$U(A,B) := \{\phi \,\big|\, \phi^{-1}\{0\} = A \wedge \phi^{-1}\{1\} = B\}$$
$$\operatorname{dom} U = \{(A,B) \,\big|\, A \cap B = \emptyset\}$$

*is $([\delta_{\Pi^0_1}, \delta_{\Pi^0_1}], [\delta \to \rho|^{[0,1]}])$-computable, for Cauchy representation $\delta$ of $X$.*

??[**?**, §22 Thm IV.1,§27 Thm II.1]
$Z := [0,1]^{\mathbb{N}}$ has metric $\rho(\xi,\eta) := \sum_{i\in\mathbb{N}} 2^{-i-1}|\xi_i - \eta_i|$.

**Lemma 6.** *Take $X \subseteq Z$, disjoint $A, B \in \Pi^0_1(X)$ and $\phi : X \to [0,1]$ continuous s.t. $\phi^{-1}\{0\} = A \wedge \phi^{-1}\{1\} = B$.*

$$f : X \to Z, x \mapsto (\phi(x), \phi(x)(1-\phi(x)).x_0, \phi(x)(1-\phi(x)).x_1, \dots)$$

*Then $f^{-1}\{a\} = A \wedge f^{-1}\{b\} = B$ for $a := (0,0,\dots,)$, $b := (1,0,0,\dots)$ and $f|_{X\setminus(A\cup B)}$ an embedding onto $f(X)\setminus\{a,b\}$.*

**Theorem 7.** *Let $(\mathcal{Y}, \delta_{\mathcal{Y}}) = (\mathcal{Z}_C(X), \delta'_{\text{disj-cover}})$. If $X \subseteq Z$ or $X$ recursively compact then $N''$ computable.*

**Proof:** If $X$ not computable subspace of $Z$, take $f' = f \circ h \circ g$ where $g := \mathrm{id}_X : (X, d) \to (X, d')$ for $d'(x, y) := \min\{d(x, y), 1\}$, $h : (X, d') \to Z, x \mapsto (d'(x, \nu(i)))_{i \in \mathbb{N}}$.

$g$ homeomorphism, $h$ embedding as in Kuratowski. $(X, d', \nu)$ computable metric space, $g, h$ computable, $f \in C(Z, Z)$ computable in $\delta_{\Pi^0_1(Z)}$-names of $(h \circ g)(A), (h \circ g)(B)$. As $X$ is $\delta_{\text{cover}}$-computable, from $\delta_{\Pi^0_1}$-names of $A, B \subseteq X$ we get $\delta_{\text{cover}}$-names, so $\delta_{\text{cover}}$-names of $(h \circ g)(A), (h \circ g)(B)$.

So $f \in C(Z, Z)$ and $f' \in C(X, Z)$ computable in $\delta_{\Pi^0_1(X)}$-names of $A, B$. If instead $X \subseteq Z$, take $f' = f \in C(X, Z)$ as in Lemma 6.

From $[\delta_X \to \delta_Z]$-name of $f'$ and $\delta'_{\text{disj-cover}}$-name of $Y \in \mathcal{Y}$, obtain a $\delta_{\text{cover}}$-name of $f'(Y)$ (recall Lemma 2).
$f'(Y) \cup \{a, b\} = f'(Y \setminus (A \cup B)) \cup \{a, b\}$ zero-dimensional (a point cannot raise inductive dimension of a nonempty space), with a $\delta'_{\text{disj-cover}}$-name $p$ available.

For ideal cover $u \in \mathbb{N}^*$, *formal diameter* is

$$D\langle u\rangle := \max_{i,j<|u|} d(\nu(\pi_1 u_i), \nu(\pi_1 u_j)) + \nu_{\mathbb{Q}^+}(\pi_2 u_i) + \nu_{\mathbb{Q}^+}(\pi_2 u_j).$$

Fix $\mathbb{Q}^+ \ni D < \rho(a,b) = 2^{-1}$, $k$ s.t. $p_k = 1 + \langle\langle w^{(0)}\rangle, \ldots, \langle w^{(n-1)}\rangle\rangle$ has each ideal cover $w^{(k)}$ of formal diameter $\leq D$.

Find $i, j < n$ s.t. $a, b$ formally included in $w^{(i)}, w^{(j)}$ respectively. $i$, $j$ unique (formal disjointness), with $i \neq j$ ($D < \rho(a,b)$). Write $[0,n) = F_0 \dot\cup F_1$ separating $i, j$, e.g. $F_0 = [0, \min\{i,j\}]$, $F_1 = (\min\{i,j\}, n)$.

For $U_l := \cup_{k \in F_l} U(w^{(k)})$ note $a \in U_0 \wedge b \in U_1 \wedge f'(Y) \subseteq U_0 \dot\cup U_1$. For $U := (f')^{-1}U_0$, $V := (f')^{-1}U_1$ then

$$A \subseteq U \wedge B \subseteq V \wedge Y \subseteq (f')^{-1}f'(Y) \subseteq U \dot\cup V,$$

with $\delta_{\Sigma^0_1(X)}$-names of $U, V$ available. $\square$

## Case $X = \mathbb{B}$

Consider map $f : X \to Y$. Any $m : X \times \mathbb{N} \rightrightarrows \mathbb{N}$ s.t.

$$(\forall x \in X)(\forall k)(\forall l)\left(l \in m(x,k) \implies B(fx; 2^{-l}) \subseteq fB(x; 2^{-k})\right)$$

is a *modulus of openness for* $f$. For any $U \in \Sigma^0_1(X)$, any choice of $k_x, l_x \in \mathbb{N}$ s.t. $B(x; 2^{-k_x}) \subseteq U \wedge l_x \in m(x, k_x)$ ($x \in U$) gives $f(U) = \cup_{x \in U} B(fx; 2^{-l_x})$.

[?]

**Theorem 8.** *If $X$, $Y$ computable metric spaces, $f : X \to Y$ computable, $m : X \times \mathbb{N} \rightrightarrows \mathbb{N}$ a computable modulus of openness for $f$ then $f$ is computably open, i.e. $\Sigma^0_1(X) \to \Sigma^0_1(Y), U \mapsto f(U)$ well-defined & computable.*

$X := \mathbb{B}$, $\ell(p,q) := \inf\{i \in \mathbb{N} \mid p_i \neq q_i\} \in \bar{\mathbb{N}}$, $d(p,q) := 2^{-\ell(p,q)}$, $\nu : \mathbb{N} \to \mathbb{N}^*.0^\omega \subseteq \mathbb{B}, \langle w\rangle \mapsto w.0^\omega$, $f = h$, $Y = h\mathbb{B} \subseteq [0,1]^{\mathbb{N}}$ as in proof of Theorem 7. We check Theorem 8 applies with $m : \mathbb{B} \times \mathbb{N} \to \mathbb{N}, (p,n) \mapsto 1 + n + \langle p^n\rangle$:

**Lemma 9.** *For $a,b,c \in \mathbb{Z}$, $|2^{-a} - 2^{-b}| < 2^{-c}$ iff $a = b \vee \min\{a,b\} \geq c$*

If $\rho(hp,hq) = \sum_{i\in\mathbb{N}} 2^{-i-1}|2^{-\ell(p,\nu(i))} - 2^{-\ell(q,\nu(i))}| < 2^{-k}$, $k \in \mathbb{N}$ then $\ell(p,\nu(i)) = \ell(q,\nu(i)) \vee \min\{\ell(p,\nu(i)), \ell(q,\nu(i))\} \geq k - i - 1$ for all $i \in \mathbb{N}$. Equivalently,

$$\bigwedge_{\langle w\rangle < k-1} \begin{pmatrix} (\exists m)(p^m = q^m = (w.0^\omega)^m \wedge p_m \neq (w.0^\omega)_m \neq q_m) \vee \\ p^{k-\langle w\rangle -1} = q^{k-\langle w\rangle -1} = (w.0^\omega)^{k-\langle w\rangle -1} \end{pmatrix}.$$

Given $p \in \mathbb{B}$, $n \in \mathbb{N}$ pick $k > n + \langle p^n\rangle$. Any $q \in h^{-1}B_\rho(hp; 2^{-k})$ has (for $w := p^n$) $p^n = q^n = w$: in 1st case $m \geq |w|$, in 2nd case $k - \langle w\rangle - 1 \geq n$. So $h^{-1}B_\rho(hp; 2^{-k}) \subseteq B_d(p; 2^{-n})$.

$h : X \to Y$ homeomorphism, so $h$ computably closed.

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